

Derived categories of noncommutative quadrics and Hilbert schemes of points

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Abstract

A non-commutative deformation of quadric surface is usually taken to be a three-dimensional cubic Artin–Schelter regular algebra. In this paper we show that for such an algebra its bounded derived category embeds into the bounded derived category of a commutative deformation of the Hilbert scheme of two points on the quadric. This is the second example in support of a conjecture by Orlov. Based on this example we formulate an infinitesimal version of the conjecture, and provide some evidence in the case of smooth projective surfaces.

Contents

1	Introduction	2
2	The geometry of $\mathrm{Gr}(1, 3)$ and $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$	4
2.1	Grassmannians	4
2.2	Hilbert schemes of points	5
2.3	The derived category of $\mathrm{Gr}(1, 3)$ and Orlov’s blow-up formula . .	8
3	Embedding derived categories of noncommutative quadrics	9
3.1	Geometric squares and deformations of $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$	9
3.2	Noncommutative quadrics	13
4	Further remarks	18

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1 Introduction

In this paper we study the derived category of a quadric (and its noncommutative analogues) in relationship with the derived category of the Hilbert scheme of two points on a quadric (and commutative deformations thereof). The motivation comes from several seemingly disparate observations.

First of all let S be a smooth projective surface (over an algebraically closed field k throughout). Then it is a classical result of Fogarty that the Hilbert scheme of n points $\text{Hilb}^n S$ is again a smooth projective variety, of dimension $2n$ [6]. If we moreover assume that $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$ (e.g. S is a rational surface, such as a quadric) then Krug–Sosna [11] have proven that the Fourier–Mukai functor

$$\Phi_{\mathcal{I}_{\mathcal{U}_n}} : \mathbf{D}^b(S) \rightarrow \mathbf{D}^b(\text{Hilb}^n S) \quad (1)$$

is a fully faithful functor, where $\mathcal{I}_{\mathcal{U}_n}$ is the ideal sheaf for the universal family $\mathcal{U}_n \subset S \times \text{Hilb}^n S$.

Another piece of motivation stems from the notion of geometric dg categories as introduced by Orlov [17]. He shows that any dg category whose homotopy category has a full exceptional collection can be embedded in (an enhancement of) the derived category of a smooth projective variety. This construction can be applied to the full exceptional collection describing the derived category of a quadric surface, but the resulting variety is constructed using iterated projective bundles and does not seem to have a geometric interpretation in terms of a moduli problem, unlike the embedding (1).

It is interesting to try and apply Orlov’s algorithm to a dg category with a full exceptional collection which is not of geometric origin. This brings us to the final piece of motivation: deformations of abelian categories. In noncommutative algebraic geometry a central role is played by abelian categories and their derived categories, and there is a framework for describing the deformations of abelian categories [13, 14], so in particular it can be applied to $\text{coh } \mathbb{P}^1 \times \mathbb{P}^1$. The quadric is easily seen to be rigid (i.e. $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1}) = 0$), but its category of coherent sheaves has nontrivial deformations ($\text{HH}^2(\text{coh } \mathbb{P}^1 \times \mathbb{P}^1) \cong H^0(\mathbb{P}^1 \times \mathbb{P}^1, \bigwedge^2 \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1})$ is 10-dimensional), which can be seen as deformations of $\mathbb{P}^1 \times \mathbb{P}^1$ in the noncommutative direction.

Because the quadric has a strongly ample sequence it is moreover possible to pass from infinitesimal deformations to formal deformations [23], and the theory has been worked out in detail in [20, 22]. A noncommutative quadric is an abelian category $\text{qgr } A$, which is a certain quotient category of the category of graded modules for a (generalized) graded algebra A satisfying some natural conditions. On the derived level it is possible to view a family of noncommutative quadrics by varying the relations in the quiver coming from the full and strong exceptional collection [15]. For these new exceptional collections it makes sense to apply Orlov’s embedding result, but again the result is an iterated projective bundle

construction where arbitrary choices have been made and there is no moduli interpretation.

Yet for \mathbb{P}^2 (and its noncommutative deformations) Orlov shows that there exists an embedding in (a commutative deformation of) $\text{Hilb}^2 \mathbb{P}^2$ [18], hence there *is* a moduli interpretation for the derived category of the finite-dimensional algebras whose structure resembles that of the Beilinson quiver for \mathbb{P}^2 .

In this paper we obtain a result analogous to Orlov’s for noncommutative quadrics. The following is a compressed version of theorem 26 and is our main result.

Theorem 1. For a generic noncommutative quadric A there exists a deformation H of $\text{Hilb}^2 \mathbb{P}^1 \times \mathbb{P}^1$ and a fully faithful embedding

$$\mathbf{D}^b(\text{qgr } A) \hookrightarrow \mathbf{D}^b(\text{coh } H). \quad (2)$$

To prove this result we need an explicit geometric model for $\text{Hilb}^2 \mathbb{P}^1 \times \mathbb{P}^1$, which we give in proposition 3. In section 3.1 we explain how this geometric model depends on so called geometric squares: linear algebra data that describes the composition law in the derived category. In section 3.2 it is shown how a sufficiently generic noncommutative quadric gives rise to such a geometric square, and indeed to an embedding as in theorem 1.

Note that there exists a notion of Hilbert scheme of points for a general cubic Artin–Schelter regular graded algebra [3], which is a subset of all noncommutative quadrics. We do not address the comparison between these moduli spaces and the deformations constructed in this paper.

We also formulate a general question regarding limited functoriality of Hochschild cohomology and the Hochschild–Kostant–Rosenberg decomposition, motivated by a conjecture of Orlov. This is done in section 4. We discuss some evidence suggesting an interesting relationship between the Hochschild cohomology of a surface and the Hochschild cohomology of the Hilbert scheme of points, showing that the results in this paper hint towards a much more general picture.

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2 The geometry of $\mathrm{Gr}(1, 3)$ and $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$

Throughout the paper, we will assume k is an algebraically closed field of characteristic 0.

2.1 Grassmannians

For a vector space V of dimension $n + 1$, denote by $\mathbb{P}(V) = \mathbb{P}^n$ the projective space of all hyperplanes of V . Denote by $G := \mathrm{Gr}(k, n)$ the Grassmannian of k -dimensional linear subspaces in \mathbb{P}^n . This is naturally identified with the set of $(k + 1)$ -dimensional linear subspaces of V^\vee , or $(n - k - 1)$ -dimensional quotients of V . The Grassmannian of k -planes in \mathbb{P}^n is naturally identified with the Grassmannian of $(n - k - 1)$ -planes in $(\mathbb{P}^n)^\vee$. We say there is a duality among the Grassmannians $\mathrm{Gr}(k, n)$ and $\mathrm{Gr}(n - k - 1, n)$.

There is a tautological exact sequence of vector bundles

$$0 \rightarrow \mathcal{R} \rightarrow V \otimes \mathcal{O}_G \rightarrow \mathcal{Q} \rightarrow 0, \quad (3)$$

where $\mathrm{rk}(\mathcal{R}) = n - k$, $\mathrm{rk}(\mathcal{Q}) = k + 1$. Also, there are identifications $H^0(G, \mathcal{Q}) \cong V$ and (by considering the dual Grassmannian), $H^0(G, \mathcal{R}) \cong V^\vee$. The Grassmannian is a fine moduli space (with universal object \mathcal{Q}) for the functor $F_G: \mathrm{Sch}^{\mathrm{op}} \rightarrow \mathrm{Sets}$ sending a scheme X to the set of epimorphisms $V \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F}$, where \mathcal{F} is a rank $k + 1$ vector bundle on X . In particular there is a bijection

$$\mathrm{Hom}_{\mathrm{Sch}}(X, G) \rightarrow F_G(X), \quad (4)$$

given by pulling back the universal epimorphism from (3) along a morphism in $\mathrm{Hom}(X, G)$. In the other direction, given an epimorphism $V \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F}$, we define an element of $\mathrm{Hom}_{\mathrm{Sch}}(X, G)$ by sending an $x \in X$ to the element of G defined by the induced map on the fibres $V \twoheadrightarrow \mathcal{F}_x \otimes k(x)$.

From now on we will focus on a specific case. Let V denote a 4-dimensional vector space, and take $k = 1$. Then $\mathbb{G} := \mathrm{Gr}(1, 3)$ is the Grassmannian of lines in \mathbb{P}^3 .

The following lemma will be used in section 3.1 to construct strong exceptional collections.

Lemma 2. Let V_1, V_2 be 2-dimensional vector spaces. Then with V as above, any isomorphism $\phi: V \rightarrow V_1 \otimes V_2$ induces a factorization

$$\mathcal{R} \xrightarrow{V_2^\vee} \ker(H^0(\mathbb{G}, \mathcal{O}_L(1)) \otimes \mathcal{O}_{\mathbb{G}} \rightarrow \mathcal{O}_L(1)) \xrightarrow{V_1^\vee} \mathcal{O}_{\mathbb{G}} \quad (5)$$

of the canonical map $\mathcal{R} \xrightarrow{V^\vee} \mathcal{O}_{\mathbb{G}}$ from (3), where the embedding $L = \mathbb{P}(V_1) \hookrightarrow \mathbb{G}$ is induced by the epimorphism $H^0(L, V_2 \otimes \mathcal{O}_L(1)) \otimes \mathcal{O}_L \twoheadrightarrow V_2 \otimes \mathcal{O}_L(1)$.

Proof. First notice that $V \cong H^0(L, V_2 \otimes \mathcal{O}_L(1))$, so by (4) the epimorphism does indeed induce an embedding $L = \mathbb{P}(V_1) \hookrightarrow \mathbb{G}$. This embedding can be explicitly described as follows: a $p \in L$ (i.e. a linear functional $p: V_1 \rightarrow k$) determines a linear map $f_p: V \rightarrow V_2$, which is just contraction by p , defining a closed point of \mathbb{G} .

Now we check that the factorization

$$\mathcal{R} \xrightarrow{V_2^\vee} V_1 \otimes \mathcal{O}_{\mathbb{G}} \xrightarrow{V_1^\vee} \mathcal{O}_{\mathbb{G}} \quad (6)$$

of the canonical map $\mathcal{R} \xrightarrow{V^\vee} \mathcal{O}_{\mathbb{G}}$ can be described more precisely as

$$\mathcal{R} \xrightarrow{V_2^\vee} \ker(H^0(\mathbb{G}, \mathcal{O}_L(1)) \otimes \mathcal{O}_{\mathbb{G}} \rightarrow \mathcal{O}_L(1)) \xrightarrow{V_1^\vee} \mathcal{O}_{\mathbb{G}}. \quad (7)$$

If the image of \mathcal{R} would not land in the kernel of the evaluation morphism, then there is some point $p \in L$ where the composition

$$\mathcal{R}_p \otimes k(p) \rightarrow V_1 \xrightarrow{p} k \quad (8)$$

in the fibers is non-zero. However, by definition of the embedding of L into \mathbb{G} , we see that $\mathcal{R}_p \otimes k(p) = \ker(f_p)$, so the composition (8) is zero and we obtain a contradiction. \square

2.2 Hilbert schemes of points

The Hilbert scheme is a classical object in algebraic geometry, parametrising closed subschemes of a projective scheme. One can associate a Hilbert polynomial to a closed subscheme, and this gives rise to a disjoint union decomposition of the Hilbert scheme. In particular, for the constant Hilbert polynomial we get the *Hilbert scheme of points*.

For a smooth projective curve C one has that $\text{Hilb}^n C = \text{Sym}^n C$, in particular it is again smooth projective and of dimension n . For a smooth projective surface S it can be shown that $\text{Hilb}^n S$ is again smooth projective and of dimension $2n$. For higher-dimensional varieties and $n \gg 2$ the Hilbert scheme becomes (very) singular.

We will identify $\mathbb{P}^1 \times \mathbb{P}^1$ with its image under the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 : ([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1], \quad (9)$$

which we denote by Q , a smooth quadric surface. This surface has two rulings, and every line on Q defines a point of \mathbb{G} . We denote

$$L := L_1 \sqcup L_2 = \{l \in \mathbb{G} \mid l \subset Q\} \subset \mathbb{G}, \quad (10)$$

where L_1 (respectively L_2) corresponds to the lines in the first (respectively second) ruling. Note that $L_1 \cap L_2 = \emptyset$, and each of these two lines determines a factorization of V as in lemma 2.

The following proposition provides our main model for working with the Hilbert scheme $\mathbb{H} := \text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ of two points on $\mathbb{P}^1 \times \mathbb{P}^1$. A reference for this description is [19, theorem 1.1].

Proposition 3. There is an isomorphism $\mathbb{H} \cong \text{Bl}_L \mathbb{G}$.

Proof. Using the Segre embedding (9) there exists a surjective morphism

$$f: \mathbb{H} \rightarrow \mathbb{G}: [Z] \mapsto l_{[Z]} \quad (11)$$

where the line $l_{[Z]}$ for a point $[Z] \in \mathbb{H}$ is defined to be the line through the two points if $[Z]$ corresponds to two distinct points, otherwise we use the tangent vector to define the line.

On the open set $\mathbb{G} \setminus L$ this is a bijection, the inverse being given by the morphism mapping a line in \mathbb{P}^3 to its intersection with the quadric.

On the closed set $L \subseteq \mathbb{G}$ the fiber over $l \in L$ can be identified with \mathbb{P}^2 : it is formed by the set of pairs of points on the line l , hence $\mathbb{H}_l \cong \text{Sym}^2 \mathbb{P}^1 \cong \mathbb{P}^2$.

By the uniqueness property of blow ups, see e.g. [7, §4.6], we get the proposed isomorphism. \square

Remark 4. In [18] the embedding of a noncommutative \mathbb{P}^2 into the derived category of a deformation of $\text{Hilb}^2 \mathbb{P}^2$ is based on the description of the Hilbert scheme as $\text{Hilb}^2 \mathbb{P}^2 \cong \mathbb{P}(\text{Sym}^2 \mathcal{T}_{\mathbb{P}^2}(-1)^\vee)$.

To find an exceptional collection on \mathbb{H} that is compatible with deformations we need to describe some bundles on \mathbb{G} and on \mathbb{H} more explicitly.

Lemma 5. There are isomorphisms

$$\begin{aligned} \mathcal{Q}|_{L_i} &\cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}, \\ \mathcal{R}|_{L_i} &\cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, \\ \mathcal{N}_{L_i} \mathbb{G} &\cong \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}. \end{aligned} \quad (12)$$

Proof. The first two isomorphisms follows from (4) since the L_i are embedded using the exact sequence

$$0 \rightarrow \mathcal{O}_{L_i}(-1)^{\oplus 2} \rightarrow V \otimes \mathcal{O}_{L_i} \rightarrow \mathcal{O}_{L_i}(1)^{\oplus 2} \rightarrow 0 \quad (13)$$

as in lemma 2.

For the third isomorphism, since the tangent bundle $\mathcal{T}_{\mathbb{G}}$ can be expressed as $\mathcal{H}om(\mathcal{R}, \mathcal{Q})$, we get for the normal bundle

$$\begin{aligned} \mathcal{N}_{L_i} \mathbb{G} &= \text{coker}(\mathcal{T}_{\mathbb{P}^1} \rightarrow \mathcal{H}om(\mathcal{R}, \mathcal{Q})|_{\mathbb{P}^1}) \\ &= \text{coker}(\mathcal{O}_{\mathbb{P}^1}(2) \rightarrow \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 4}) \\ &= \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}. \end{aligned} \quad (14)$$

\square

From the description of the normal bundle $\mathcal{N}_{L_i}\mathbb{G}$ in lemma 5 we find that

$$E_i \cong \mathbf{Proj}_{\mathbb{P}^1}(\mathrm{Sym} \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}) \cong \mathbb{P}^1 \times \mathbb{P}^2. \quad (15)$$

We will use the following notation

$$\mathcal{O}_{E_i}(m, n) := \mathcal{O}_{\mathbb{P}^1}(m) \boxtimes \mathcal{O}_{\mathbb{P}^2}(n). \quad (16)$$

Whenever we write $\mathcal{O}_E(m, n)$ this means that we use this construction for both connected components.

For the final lemma, recall that $\mathcal{O}_E(E)$ is shorthand for $\mathcal{O}_{\mathbb{H}}(E)|_E = j^*(\mathcal{O}_{\mathbb{H}}(E))$, which can also be written as $\mathcal{N}_E\mathbb{H}$. Using this notation we can describe $\omega_{\mathbb{H}}$ and two bundles on the exceptional locus E as follows.

Lemma 6. There are isomorphisms

$$\omega_{\mathbb{H}} \cong p^* \left(\bigwedge^2 \Omega \right)^{\otimes -4} (2E), \quad (17)$$

and

$$\begin{aligned} \mathcal{O}_E(E) &\cong \mathcal{O}_E(2, -1), \\ \omega_{\mathbb{H}}|_E &\cong \mathcal{O}_E(-4, -2). \end{aligned} \quad (18)$$

Proof. Applying the adjunction formula and the isomorphism $\mathcal{N}_{L_i}\mathbb{G} \cong \mathcal{O}_{\mathbb{P}^1}(2)^{\oplus 3}$ from lemma 5, we find

$$\begin{aligned} \omega_{\mathbb{P}^1} &\cong i^*(\omega_{\mathbb{G}}) \otimes \det(\mathcal{N}_{L_i}\mathbb{G}) \\ &\Leftrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \cong \omega_{\mathbb{G}}|_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(6) \\ &\Leftrightarrow \mathcal{O}_{\mathbb{P}^1}(-8) \cong \omega_{\mathbb{G}}|_{\mathbb{P}^1}. \end{aligned} \quad (19)$$

For the canonical bundles, we get

$$\omega_{\mathbb{H}} \cong p^*(\omega_{\mathbb{G}}) \otimes \mathcal{O}_{\mathbb{H}}(2E), \quad (20)$$

and

$$\omega_E \cong (\omega_{\mathbb{H}} \otimes \mathcal{O}_{\mathbb{H}}(E))|_E. \quad (21)$$

Now plug (20) into (21) and use $\omega_{E_i} \cong \mathcal{O}_{E_i}(-2, -3)$ to get

$$\begin{aligned} \mathcal{O}_E(-2, -3) &\cong \mathcal{O}_E(-8, 0) \otimes \mathcal{O}_E(3E) \\ &\Leftrightarrow \mathcal{O}_E(6, -3) \cong \mathcal{O}_E(3E) \\ &\Leftrightarrow \mathcal{O}_E(E) \cong \mathcal{O}_E(2, -1). \end{aligned} \quad (22)$$

Finally (20) provides

$$\omega_{\mathbb{H}}|_E \cong \mathcal{O}_E(-8, 0) \otimes \mathcal{O}_E(4, -2) \cong \mathcal{O}_E(-4, -2), \quad (23)$$

completing the proof. \square

2.3 The derived category of $\mathrm{Gr}(1, 3)$ and Orlov's blow-up formula

Based on proposition 3 we will use the following notation for the rest of the paper.

$$\begin{array}{ccc}
 E = E_1 \sqcup E_2 & \xrightarrow{j = j_1 \sqcup j_2} & \mathbb{H} := \mathrm{Bl}_{L_1 \sqcup L_2} \mathbb{G} \\
 \downarrow q = q_1 \sqcup q_2 & & \downarrow p \\
 L = L_1 \sqcup L_2 & \xleftarrow{i = i_1 \sqcup i_2} & \mathbb{G}.
 \end{array} \tag{24}$$

The following theorem is a particular case of a more general result obtained in [2, 9].

Theorem 7. The derived category of \mathbb{G} has a full and strong exceptional collection

$$\mathbf{D}^b(\mathbb{G}) = \left\langle \bigwedge^2 \mathcal{R} \otimes \bigwedge^2 \mathcal{R}, \bigwedge^2 \mathcal{R} \otimes \mathcal{R}, \bigwedge^2 \mathcal{R}, \mathrm{Sym}^2 \mathcal{R}, \mathcal{R}, \mathcal{O}_{\mathbb{G}} \right\rangle. \tag{25}$$

Remark 8. In fact, we will only need the exceptional pair $\langle \mathcal{R}, \mathcal{O}_{\mathbb{G}} \rangle$, which can also be established by elementary means.

We know from proposition 3 that $\mathbb{H} \cong \mathrm{Bl}_L(\mathbb{G})$, so the following classical result of Orlov describes the derived category of \mathbb{H} . Let Y be a smooth subvariety of codimension r in a smooth algebraic variety X . Then there exists a cartesian square

$$\begin{array}{ccc}
 E_Y & \xrightarrow{j} & \mathrm{Bl}_Y X \\
 \downarrow q & & \downarrow p \\
 Y & \xrightarrow{i} & X
 \end{array} \tag{26}$$

where i and j are closed immersions, and $q: E_Y \rightarrow Y$ is the projective bundle of the exceptional divisor E_Y in $\mathrm{Bl}_Y X$ on Y .

Theorem 9. [16, theorem 4.3] There is a semi-orthogonal decomposition

$$\mathbf{D}^b(\mathrm{Bl}_Y X) = \langle \mathbf{D}^b(X), \mathbf{D}^b(Y)_1, \dots, \mathbf{D}^b(Y)_{r-1} \rangle. \tag{27}$$

In this statement, $\mathbf{D}^b(X)$ is the full subcategory of $\mathbf{D}^b(\mathrm{Bl}_Y X)$ which is the image of $\mathbf{D}^b(X)$ under

$$\mathbf{L}p^*: \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(\mathrm{Bl}_Y X), \tag{28}$$

and $\mathbf{D}^b(Y)_k$ is the full subcategory of $\mathbf{D}^b(\mathrm{Bl}_Y X)$ which is the image of $\mathbf{D}^b(Y)$ under

$$\mathbf{R}j_*(\mathcal{O}_{E_Y}(k) \otimes q^*(-)) : \mathbf{D}^b(Y) \rightarrow \mathbf{D}^b(\mathrm{Bl}_Y X). \tag{29}$$

Corollary 10. There is a semi-orthogonal decomposition

$$\begin{aligned} \mathbf{D}^b(\mathbb{H}) &= \left\langle \mathbf{D}^b(\mathbb{G}), \mathbf{D}^b(L)_1, \mathbf{D}^b(L)_2 \right\rangle \\ &= \left\langle \mathbf{D}^b(\mathbb{G}), \mathbf{D}^b(L_1)_1, \mathbf{D}^b(L_1)_2, \mathbf{D}^b(L_2)_1, \mathbf{D}^b(L_2)_2 \right\rangle \end{aligned} \quad (30)$$

In particular there exists a full exceptional collection of length 14 in $\mathbf{D}^b(\mathbb{H})$.

Remark 11. This is not the only way of obtaining a semi-orthogonal decomposition of the Hilbert scheme in this situation. For an arbitrary surface S one obtains using equivariant derived categories [5] and the description of the Hilbert scheme of points as a quotient that there exists a full (and strong) exceptional collection in $\mathbf{D}^b(\text{Hilb}^n S)$ provided there exists a full (and strong) exceptional collection in $\mathbf{D}^b(S)$ [11, proposition 1.3 and remark 4.6].

3 Embedding derived categories of noncommutative quadrics

3.1 Geometric squares and deformations of $\text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$

Recall [9] that for the derived category of the quadric Q there is a full and strong exceptional collection

$$\begin{array}{ccccc} & & \mathcal{O}_Q(-1, 0) & & \\ & \nearrow^{c_2} & & \nwarrow_{d_1} & \\ \mathcal{O}_Q(-1, -1) & & & & \mathcal{O}_Q(0, 0) \\ & \searrow_{c_1} & & \nearrow_{d_2} & \\ & & \mathcal{O}_Q(0, -1) & & \end{array} \quad (31)$$

$\begin{array}{ccccc} & & \mathcal{O}_Q(0, -1) & & \\ & \nearrow_{a_1} & & \nwarrow_{b_1} & \\ \mathcal{O}_Q(-1, -1) & & & & \mathcal{O}_Q(0, 0) \\ & \searrow_{a_2} & & \nearrow_{b_2} & \end{array}$

with relations $b_i a_j = d_j c_i$, for $i, j \in \{1, 2\}$. We isolate some of the properties of this exceptional collection in the following definition.

Definition 12. A *geometric square* is a septuple $\square = (V, U_0^0, U_1^0, U_0^1, U_1^1, \phi_0, \phi_1)$, where V is a 4-dimensional vectorspace, the U_i^j are 2-dimensional vector spaces, and the ϕ_i are isomorphisms

$$\phi_i: V \rightarrow U_0^i \otimes U_1^i. \quad (32)$$

Using lemma 2, the two isomorphisms ϕ_i in a geometric square give rise to two embeddings $L_i := \mathbb{P}(U_0^i) \hookrightarrow \mathbb{G}$ and sheaves

$$K_i := \ker \left(H^0(\mathbb{G}, \mathcal{O}_{L_i}(1)) \otimes \mathcal{O}_{\mathbb{G}} \rightarrow \mathcal{O}_{L_i}(1) \right). \quad (33)$$

Proposition 13. For a sufficiently generic geometric square \square , the Ext-quiver of the endomorphism algebra

$$Q_\square := \text{End}_{\mathbb{G}}(\mathcal{R} \oplus K_0 \oplus K_1 \oplus \mathcal{O}_{\mathbb{G}}) \quad (34)$$

is of the form (31), and moreover

$$\dim \text{Hom}(\mathcal{R}, \mathcal{O}_{\mathbb{G}}) = 4. \quad (35)$$

Proof. We first check that there are no Hom's going backwards. Applying $\text{Hom}(-, \mathcal{R})$ to (33) we see that by exceptionality of the pair $\langle \mathcal{R}, \mathcal{O}_{\mathbb{G}} \rangle$ we need to prove that $\text{Ext}^1(\mathcal{O}_{L_i}(1), \mathcal{R}) = 0$. This is the case by Serre duality:

$$\begin{aligned} \text{Ext}_{\mathbb{G}}^1(\mathcal{O}_{L_i}(1), \mathcal{R}) &\cong \text{Ext}_{\mathbb{G}}^3(\mathcal{R} \otimes \omega_{\mathbb{G}}^\vee, \mathcal{O}_{L_i}(1))^\vee \\ &\cong \text{Ext}_{L_i}^3((\mathcal{R} \otimes \omega_{\mathbb{G}}^\vee)|_{L_i}, \mathcal{O}_{L_i}(1))^\vee \\ &= 0. \end{aligned} \quad (36)$$

Applying $\text{Hom}(\mathcal{O}_{\mathbb{G}}, -)$ to (33) we get that K_i indeed does not have global sections because we get the identity morphism between $\text{Hom}(\mathcal{O}_{\mathbb{G}}, H^0(\mathbb{G}, \mathcal{O}_{L_i}(1)) \otimes \mathcal{O}_{\mathbb{G}})$ and $\text{Hom}(\mathcal{O}_{\mathbb{G}}, \mathcal{O}_{L_i}(1))$.

Now to each of the isomorphisms ϕ_i we can apply lemma 2, and for a generic geometric square, the $\mathbb{P}(U_0^i)$ don't intersect in \mathbb{G} , hence $\text{Hom}(K_i, K_{1-i}) = 0$, and the algebra Q_\square does indeed have the form (31). \square

These four coherent sheaves cannot be used to realize an admissible embedding $\mathbf{D}^b(Q_\square) \hookrightarrow \mathbf{D}^b(\mathbb{G})$ since they do not form an exceptional collection. To ensure that they do, we need to blow up \mathbb{G} in the two L_i , mimicking the description in proposition 3. Let us denote by E_i the corresponding exceptional divisors on $\mathbb{H}_\square := \text{Bl}_{L_0 \sqcup L_1} \mathbb{G}$, so we have a cartesian square

$$\begin{array}{ccc} E_\square = E_0 \sqcup E_1 & \xrightarrow{j = j_1 \sqcup j_2} & \mathbb{H}_\square = \text{Bl}_{L_0 \sqcup L_1} \mathbb{G} \\ \downarrow q = q_1 \sqcup q_2 & & \downarrow p \\ L_\square = L_0 \sqcup L_1 & \xleftarrow{i = i_1 \sqcup i_2} & \mathbb{G} \end{array} \quad (37)$$

similar to (24).

We are now ready to show how a generic geometric square gives rise to a strong exceptional collection of vector bundles. In theorem 15 we will describe the structure of this strong exceptional collection.

In the proof we will compute mutations of exceptional collections. If $\langle E, F \rangle$ is an exceptional collection we will denote the left mutated collection as $\langle L_E F, E \rangle$. A special property of the exceptional collection in (31) is that it is a 3-block collection, and one can also mutate blocks, for which similar notation will be used.

Theorem 14. For a generic geometric square, there is a strong exceptional collection of vector bundles

$$\langle p^*\mathcal{R}, C_1, C_2, \mathcal{O}_{\mathbb{H}_\square} \rangle \quad (38)$$

of ranks 2, 2, 2, 1 on \mathbb{H}_\square , where

$$C_i = \ker(\mathcal{O}_{\mathbb{H}_\square}^{\oplus 2} \twoheadrightarrow \mathcal{O}_{E_i}(1, 0)). \quad (39)$$

Proof. The first and last object are clearly vector bundles. For the middle objects, this can be checked in the fibers by tensoring the defining short exact sequence of C_i with the residue field in a point, and using that $\mathcal{O}_{E_i}(1, 0)$ is the pushforward of a line bundle on E_i , hence locally has the divisor short exact sequence as a flat resolution.

The derived pullback $\mathbf{L}p^*$ is fully faithful, and $\mathbf{L}p^* = p^*$ when applied to vector bundles. Since $\langle \mathcal{R}, \mathcal{O}_{\mathbb{G}} \rangle$ is a strong exceptional pair by theorem 7, so is $\langle p^*\mathcal{R}, \mathcal{O}_{\mathbb{H}_\square} \rangle$.

We first check that $\langle E, [F, G] \rangle = \langle \mathcal{O}_{\mathbb{H}_\square}, [\mathcal{O}_{E_0}(1, 0), \mathcal{O}_{E_1}(1, 0)] \rangle$ is a strong (block) exceptional collection. The sheaves $\mathcal{O}_{E_i}(1, 0)$ are exceptional by the fully faithfulness of (29); moreover

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_{E_i}(1, 0), \mathcal{O}_{\mathbb{H}_\square}[k]) &= \mathrm{Hom}(\mathcal{O}_{\mathbb{H}_\square}, \mathcal{O}_{E_i}(1, 0) \otimes \omega_{\mathbb{H}_\square}[4 - k])^\vee \\ &= H^{4-k}(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(-3, -2))^\vee \\ &= 0, \end{aligned} \quad (40)$$

where we used (18) in the second equality. Also,

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{H}_\square}, \mathcal{O}_{E_i}(1, 0)[k]) = H^k(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(1, 0)), \quad (41)$$

and $\mathcal{O}_{E_0}(1, 0), \mathcal{O}_{E_1}(1, 0)$ are orthogonal because they have disjoint support, so $\langle E, [F, G] \rangle$ is indeed a strong (block) exceptional collection. Hence the mutated collection

$$\langle [L_E(F), L_E(G)], E \rangle = \langle [C_0, C_1], \mathcal{O}_{\mathbb{H}_\square} \rangle \quad (42)$$

is also exceptional. By applying $\mathrm{Hom}(-, \mathcal{O}_{\mathbb{H}_\square})$ to the defining short exact sequence for C_i

$$0 \rightarrow C_i \rightarrow \mathcal{O}_{\mathbb{H}_\square}^{\oplus 2} \rightarrow \mathcal{O}_{E_i}(1, 0) \rightarrow 0, \quad (43)$$

obtained from the mutation and using that $\langle \mathcal{O}_{\mathbb{H}_\square}, \mathcal{O}_{E_i}(1, 0) \rangle$ is exceptional, we see that $\langle [C_0, C_1], \mathcal{O}_{\mathbb{H}_\square} \rangle$ is a strong exceptional collection.

It remains to check that $\langle p^*\mathcal{R}, [C_0, C_1] \rangle$ is a strong exceptional collection. We first check strongness: applying $\mathrm{Hom}(p^*\mathcal{R}, -)$ to (43) and using that $\langle p^*(\mathcal{R}), \mathcal{O}_{\mathbb{H}_\square} \rangle$ is exceptional, we find an exact sequence

$$0 \rightarrow \mathrm{Hom}(p^*\mathcal{R}, C_i) \rightarrow \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{\mathbb{H}_\square}^{\oplus 2}) \rightarrow \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{E_i}(1, 0)) \rightarrow \mathrm{Ext}^1(p^*\mathcal{R}, C_i) \rightarrow 0, \quad (44)$$

and

$$\mathrm{Ext}^{i+1}(p^*\mathcal{R}, C_i) \cong \mathrm{Ext}^i(p^*\mathcal{R}, \mathcal{O}_{E_i}(1, 0)), \quad (45)$$

for all $i \geq 1$. Now

$$\mathrm{Ext}^i(p^*\mathcal{R}, \mathcal{O}_{E_i}(1, 0)) \cong H^i(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(2, 0)^{\oplus 2}), \quad (46)$$

which is zero for $i \geq 1$. Also

$$\begin{aligned} \dim \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{\mathbb{H}_{\square}}^{\oplus 2}) &= 8, \\ \dim \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{E_i}(1, 0)) &= 6, \end{aligned} \quad (47)$$

so it suffices to note that

$$\begin{aligned} \mathrm{Hom}(p^*\mathcal{R}, C_i) &\cong \mathrm{Hom}_{\mathbb{G}}(\mathcal{R}, \mathbf{R}p_*C_i) \\ &\cong \mathrm{Hom}_{\mathbb{G}}(\mathcal{R}, K_i), \end{aligned} \quad (48)$$

which is 2-dimensional by proposition 13. Finally we check exceptionality: again one can apply $\mathrm{Hom}(-, p^*\mathcal{R})$ to (43) to see that

$$\mathrm{Ext}^i(C_i, p^*\mathcal{R}) \cong \mathrm{Ext}^{i+1}(\mathcal{O}_{E_i}(1, 0), p^*\mathcal{R}), \quad (49)$$

and this last group can be calculated using Serre duality, lemma 6 and lemma 5 as follows:

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}_{E_i}(1, 0), p^*\mathcal{R}[k+1]) &\cong \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{E_i}(1, 0) \otimes \omega_{\mathbb{H}_{\square}}[4-k-1])^{\vee} \\ &\cong \mathrm{Hom}(p^*\mathcal{R}, \mathcal{O}_{E_i}(-3, -2)[4-k-1])^{\vee} \\ &\cong H^{4-k-1}(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(-2, -2)^{\oplus 2})^{\vee}, \end{aligned} \quad (50)$$

which is easily seen to be zero for all k . \square

Theorem 15. For a generic geometric square \square , there is an admissible embedding

$$\mathbf{D}^b(Q_{\square}) \hookrightarrow \mathbf{D}^b(\mathbb{H}_{\square}), \quad (51)$$

where Q_{\square} is the endomorphism algebra as in (34), and \mathbb{H}_{\square} is a deformation of \mathbb{H} .

Proof. By theorem 14, there is an admissible embedding

$$\mathbf{D}^b(\mathrm{End}(p^*\mathcal{R} \oplus C_1 \oplus C_2 \oplus \mathcal{O}_{\mathbb{H}_{\square}})) \hookrightarrow \mathbf{D}^b(\mathbb{H}_{\square}). \quad (52)$$

Because i is a closed immersion we get the exact sequence

$$0 \rightarrow L^1 p^*(\mathcal{O}_{L_i}(1)) \rightarrow p^*(K_i) \rightarrow \mathcal{O}_{\mathbb{H}}^2 \rightarrow \mathcal{O}_{E_i}(1, 0) \rightarrow 0 \quad (53)$$

after applying $\mathbf{L}p^*$ to (33) and hence by quotienting out the torsion in $p^*(K_i)$ we obtain an isomorphism

$$C_i \cong p^*(K_i)/L^1 p^*(\mathcal{O}_{L_i}(1)). \quad (54)$$

The last thing to observe is that the action of Q_\square remains faithful, which gives us the isomorphism

$$\mathrm{End}_{\mathbb{H}}(p^*\mathcal{R} \oplus C_1 \oplus C_2 \oplus \mathcal{O}_{\mathbb{H}_\square}) \cong Q_\square \quad (55)$$

and the admissible embedding (51).

To see this, it suffices to realise that the action of Q_\square generically does not change under taking the quotient with the torsion subsheaf. So if an element of Q_\square were to act as zero on the exceptional collection on \mathbb{H} , it would also act as zero on the original collection of sheaves on \mathbb{G} because all sheaves are torsionfree, arriving at a contradiction. \square

3.2 Noncommutative quadrics

We will now recall the necessary definitions and some properties of noncommutative quadrics, all of which are proven in [22]. Then we will explain how a generic noncommutative quadric gives rise to a geometric square, such that we can prove the embedding result in theorem 26.

A \mathbb{Z} -algebra is a pre-additive category with objects indexed by \mathbb{Z} , generalising the theory of graded algebras and modules. All usual notions like right and left modules, bimodules, ideals, etc. make sense in this context and we will freely make use of them. For more details, consult [22, §2].

Let $\mathrm{Gr}(A)$ denote the category of right A -modules, and (if A is noetherian) $\mathrm{gr}(A)$ the full subcategory of noetherian objects. Also $\mathrm{QGr}(A)$ (respectively $\mathrm{qgr}(A)$) is the quotient of $\mathrm{Gr}(A)$ (respectively $\mathrm{gr}(A)$) by the torsion modules. The quotient functor is denoted $\pi: \mathrm{Gr}(A) \rightarrow \mathrm{qgr}(A)$.

We write $A_{i,j} = \mathrm{Hom}_A(j, i)$, and $e_i = i \xrightarrow{\mathrm{id}} i$, for $i, j \in \mathbb{Z}$. Then $P_i = e_i A$ are projective generators for $\mathrm{Gr}(A)$ and if A is connected, S_i will be the unique simple quotient of P_i .

Definition 16. A \mathbb{Z} -algebra A is a *three-dimensional cubic Artin–Schelter-regular algebra* if

1. A is connected,
2. $\dim A_{i,j}$ is bounded by a polynomial in $j - i$,
3. $\mathrm{pdim} S_i < \infty$ and bounded independently of i ,
4. $\sum_{j,k} \dim \mathrm{Ext}_{\mathrm{Gr} A}^j(S_k, P_i) = 1$, for every i ,
5. the minimal resolution of S_i has the form

$$0 \rightarrow P_{i+4} \rightarrow P_{i+3}^{\oplus 2} \rightarrow P_{i+1}^{\oplus 2} \rightarrow P_i \rightarrow S_i \rightarrow 0. \quad (56)$$

Using this definition, we can now define noncommutative quadrics.

Definition 17. A *noncommutative quadric* is a category of the form $\mathrm{QGr}(A)$, where A is a three dimensional cubic Artin–Schelter regular algebra.

An important subclass of the cubic Artin–Schelter regular \mathbb{Z} -algebras is given by the \mathbb{Z} -algebra associated to a cubic Artin–Schelter regular graded algebra [1]. In general one gets a \mathbb{Z} -algebra \check{B} from a \mathbb{Z} -graded algebra by setting

$$\check{B}_{i,j} := B_{j-i}. \quad (57)$$

The \mathbb{Z} -algebras obtained in this way are called 1-periodic.

The motivation for this definition comes from the following theorem. For details and unexplained terminology we refer to [22, 23].

Theorem 18. [22, theorem 1.5] Let (R, \mathfrak{m}) be a complete commutative Noetherian local ring with $k = R/\mathfrak{m}$. Any R -deformation of the abelian category $\mathrm{coh} \mathbb{P}^1 \times \mathbb{P}^1$ is of the form $\mathrm{qgr} \mathcal{A}$, where \mathcal{A} is an R -family of three-dimensional cubic Artin–Schelter regular \mathbb{Z} -algebras.

One of the main results of [22] is the classification of cubic Artin–Schelter regular \mathbb{Z} -algebras in terms of linear algebra data. We will now recall this description for use in proposition 23.

A three-dimensional cubic AS-regular algebra satisfies $A_{i,i+n} = 0$ for $n < 0$. It is generated by the $V_i = A_{i,i+1}$ and the relations are generated by the

$$R_i = \ker(V_i \otimes V_{i+1} \otimes V_{i+2} \rightarrow A_{i,i+3}), \quad (58)$$

which are of dimension two. Denote by

$$W_i = V_i \otimes R_{i+1} \cap R_i \otimes V_{i+3} \subset V_i \otimes V_{i+1} \otimes V_{i+2} \otimes V_{i+3}, \quad (59)$$

which are of dimension one. Any non-zero element of W_i is a rank two tensor, both as an element of $V_i \otimes R_{i+1}$ and as an element of $R_i \otimes V_{i+3}$. Finally, A is determined up to isomorphism by its truncation $\bigoplus_{i,j=0}^3 A_{ij}$, which motivates the following definition.

Definition 19. A quintuple (V_0, V_1, V_2, V_3, W) , where the V_i are two-dimensional vector spaces and $0 \neq W = kw \subset V_0 \otimes V_1 \otimes V_2 \otimes V_3$ is called *geometric* if for all $j \in \{0, 1, 2, 3\}$, and for all $0 \neq \phi_j \in V_j^\vee$, the tensor

$$\langle \phi_j \otimes \phi_{j+1}, w \rangle \quad (60)$$

is non-zero, where indices are taken modulo four.

In the sequel we will sometimes identify a quintuple by a non-zero element of W , and we will omit the tensor product.

From the previous discussion, it is clear how to associate a quintuple to a noncommutative quadric. In fact, this quintuple is geometric and there is the following classification theorem that tells us that it suffices to consider geometric quintuples.

Theorem 20. [22, theorem 4.31] There is an isomorphism preserving bijection between noncommutative quadrics and geometric quintuples.

By construction a noncommutative quadric has a full strong exceptional collection

$$\pi(P_3) \begin{array}{c} \xrightarrow{V_2} \\ \xleftarrow{\quad} \end{array} \pi(P_2) \begin{array}{c} \xrightarrow{V_1} \\ \xleftarrow{\quad} \end{array} \pi(P_1) \begin{array}{c} \xrightarrow{V_0} \\ \xleftarrow{\quad} \end{array} \pi(P_0) \quad (61)$$

with relations $R = W \otimes V_3^\vee$. We will use the (purely formal) notation

$$\mathcal{O}(-1, -2) \begin{array}{c} \xrightarrow{V_2} \\ \xleftarrow{\quad} \end{array} \mathcal{O}(-1, -1) \begin{array}{c} \xrightarrow{V_1} \\ \xleftarrow{\quad} \end{array} \mathcal{O}(0, -1) \begin{array}{c} \xrightarrow{V_0} \\ \xleftarrow{\quad} \end{array} \mathcal{O}(0, 0) \quad (62)$$

Example 21 (Linear quadric). We can now explain how the (commutative) quadric surface gives rise to a cubic Artin–Schelter regular \mathbb{Z} -algebra. On $\mathbb{P}^1 \times \mathbb{P}^1$ there are the line bundles

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m, n) = \mathcal{O}_{\mathbb{P}^1}(m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(n). \quad (63)$$

The following defines an ample sequence:

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(n) = \begin{cases} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, k) & \text{if } n = 2k, \\ \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k+1, k) & \text{if } n = 2k+1. \end{cases} \quad (64)$$

Put $A = \bigoplus_{i,j} \text{Hom}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-j), \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-i))$. Then $\text{coh } \mathbb{P}^1 \times \mathbb{P}^1 \cong \text{qgr } A$, and A is a 3-dimensional cubic AS-regular algebra. One may choose bases x_i, y_i for V_i such that the relations in A are given by

$$\begin{aligned} x_i x_{i+1} y_{i+2} - y_i x_{i+1} x_{i+2} &= 0 \\ x_i y_{i+1} y_{i+2} - y_i y_{i+1} x_{i+2} &= 0. \end{aligned} \quad (65)$$

The tensor $w \in W_0$ corresponding to these relations is given by

$$w = x_0 x_1 y_2 y_3 - y_0 x_1 x_2 y_3 - x_0 y_1 y_2 x_3 + y_0 y_1 x_2 x_3. \quad (66)$$

The corresponding exceptional collection has quiver

$$\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-3) \begin{array}{c} \xrightarrow{x_2} \\ \xleftarrow{y_2} \end{array} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2) \begin{array}{c} \xrightarrow{x_1} \\ \xleftarrow{y_1} \end{array} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1) \begin{array}{c} \xrightarrow{x_0} \\ \xleftarrow{y_0} \end{array} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \quad (67)$$

with relations (65), corresponding to (62).

The relationship between the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ under the Segre embedding and the \mathbb{Z} -algebra A is obtained by taking the 2-Veronese of A , giving an isomorphism

$$\left(k\langle x, y \rangle / \begin{pmatrix} x^2 y - y x^2 \\ x y^2 - y^2 x \end{pmatrix}_2 \right) \cong k[a, b, c, d] / (ad - bc), \quad (68)$$

where we described the \mathbb{Z} -algebra as a graded algebra, because in this case A is 1-periodic.

Another important class of noncommutative quadrics is given by the so called type-A cubic algebras.

Example 22 (Type-A cubic algebras). We will consider the generic class of cubic algebras from [1]. In this case the (graded) algebra A has two generators x and y and relations

$$\begin{aligned} ay^2x + byxy + axy^2 + cx^3 &= 0 \\ ax^2y + bxyx + ayx^2 + cy^3 &= 0 \end{aligned} \quad (69)$$

for $(a : b : c) \in \mathbb{P}^2 - S$, where

$$S = \{(a : b : c) \in \mathbb{P}^2 \mid a^2 = b^2 = c^2\} \cup \{(0 : 0 : 1), (0 : 1 : 0)\}. \quad (70)$$

The tensor $w \in W_0$ corresponding to these relations in the \mathbb{Z} -algebra setting is given by

$$\begin{aligned} w = & ay_0y_1x_2x_3 + by_0x_1y_2x_3 + ax_0y_1y_2x_3 + cx_0x_1x_2x_3 \\ & + ax_0x_1y_2y_3 + bx_0y_1x_2y_3 + ay_0x_1x_2y_3 + cy_0y_1y_2y_3. \end{aligned} \quad (71)$$

The corresponding full and strong exceptional collection is given by

$$A(-3) \begin{array}{c} \xrightarrow{x_2} \\ \xrightarrow{y_2} \end{array} A(-2) \begin{array}{c} \xrightarrow{x_1} \\ \xrightarrow{y_1} \end{array} A(-1) \begin{array}{c} \xrightarrow{x_0} \\ \xrightarrow{y_0} \end{array} A \quad (72)$$

with relations coming from (69).

Since our model for theorem 15 was the 3-block exceptional collection (31) and not the linear collection (67), we first have to mutate a linear exceptional collection as in (62) to a square one as in (31).

Proposition 23. The exceptional collection obtained from (62) by right mutating the first two objects is strong and has endomorphism ring

$$\begin{array}{ccccc} & & \mathcal{O}(-1, 0) & & \\ & \nearrow^{V_2^\vee} & & \searrow^R & \\ \mathcal{O}(-1, -1) & & & & \mathcal{O}(0, 0) \\ & \searrow_{V_1} & & \nearrow_{V_0} & \\ & & \mathcal{O}(0, -1) & & \end{array} \quad (73)$$

where we used the notation $\mathcal{O}(-1, 0) = R_{\mathcal{O}(-1, -1)}\mathcal{O}(-1, -2)$.

Proof. By construction the right mutation $\mathcal{O}(-1, 0)$ fits in a short exact sequence

$$0 \rightarrow \mathcal{O}(-1, -2) \rightarrow V_2^\vee \otimes_k \mathcal{O}(-1, -1) \rightarrow \mathcal{O}(-1, 0) \rightarrow 0 \quad (74)$$

because we can compute the mutation entirely in $\text{qgr } A$ as the morphism on the left is indeed a monomorphism by definition.

To see that $\text{Hom}(\mathcal{O}(-1, 0), \mathcal{O}(0, 0)) = R$ one can use the proof of [22, lemma 4.3]. By applying $\text{Hom}(-, \mathcal{O}(0, 0))$ to (74) we get a long exact sequence, which by the canonical isomorphism $A_{0,2} = V_0 \otimes V_1 = \text{Hom}(\mathcal{O}(-1, -1), \mathcal{O}(0, 0))$ corresponds to

$$0 \rightarrow \text{Hom}(\mathcal{O}(-1, 0), \mathcal{O}(0, 0)) \rightarrow V_0 \otimes V_1 \otimes V_2 \rightarrow A_{0,3} \rightarrow 0, \quad (75)$$

hence $R = \text{Hom}(\mathcal{O}(-1, 0), \mathcal{O}(0, 0))$. This also shows that the higher Ext's vanish.

Finally, to see that $\mathcal{O}(-1, 0)$ and $\mathcal{O}(0, -1)$ are completely orthogonal, we can apply $\text{Hom}(-, \mathcal{O}(0, -1))$ to (74). By the resulting long exact sequence where the isomorphism $V_1 \otimes V_2 \cong A_{1,3}$ is the only non-zero map we get the desired orthogonality. \square

We are almost in a situation where we can apply theorem 15. However, an arbitrary geometric quintuple does not give rise to a geometric square since the induced map $R \otimes V_2^\vee \rightarrow V_0 \otimes V_1$ in (73) is not necessarily an isomorphism. The next proposition describes a dense subset for which this is the case. Recall that $w \in V_0 \otimes V_1 \otimes V_2 \otimes V_3$, and we have an action of \mathbb{G}_m^4 on this space, so w can be interpreted as a point in $(\mathbb{P}^1)^4$.

Proposition 24. A generic geometric quintuple (V_0, V_1, V_2, V_3, w) gives rise to a geometric square. More precisely, for w in a Zariski open subset \mathcal{U}' of $(\mathbb{P}^1)^4$,

$$\square_w = (V_0 \otimes V_1, V_0, V_1, V_2^\vee, V_3^\vee, \text{id}, \phi_w) \quad (76)$$

is a geometric square, where $\phi_w = \langle -, w \rangle^{-1}$.

Proof. By the geometricity condition we have that the morphism

$$\langle -, w \rangle: V_2^\vee \otimes V_3^\vee \rightarrow V_0 \otimes V_1 \quad (77)$$

sends the pure tensors to nonzero elements. Assume on the other hand that this morphism is not an isomorphism. Then the kernel has to intersect the quadric cone corresponding to the pure tensors trivially in the origin. This is only possible if the kernel has dimension precisely 1, corresponding to the vanishing of the determinant. The open \mathcal{U}' is defined by the complement of this vanishing locus in $(\mathbb{P}^1)^4$. So starting from a geometric quintuple with $w \in \mathcal{U}'$ we can define the associated square (76), and $\phi_w = \langle -, w \rangle^{-1}$ is an isomorphism. \square

Example 25 (Linear quadric). For the geometric quintuple (66) it is easy to see that $w \in \mathcal{U}$, so we get an associated geometric square with exceptional collection (73), which is exactly the 3-block collection (31). Another small calculation shows that the two \mathbb{P}^1 's don't intersect so $w \in \mathcal{U}'$. As expected, the two \mathbb{P}^1 's correspond to the two rulings on $\mathbb{P}^1 \times \mathbb{P}^1$, which we used in proposition 3.

Let us denote by $\mathbb{H}_w := \mathbb{H}_{\square_w}$, for $w \in \mathcal{U}'$, and by $\text{qgr } A_w$ the associated non-commutative quadric. The following is then our main result.

Theorem 26. The varieties \mathbb{H}_w form a smooth projective family \mathcal{H} over a Zariski open $\mathcal{U} \subset \mathcal{U}'$ containing \mathbb{H} , and for each $w \in \mathcal{U}$ there is an admissible embedding

$$\mathbf{D}^b(\text{qgr } A_w) \hookrightarrow \mathbf{D}^b(\mathbb{H}_w). \quad (78)$$

by vector bundles of ranks 2, 2, 2 and 1.

Proof. This is now immediate from the combination of theorem 15, proposition 24 and proposition 23. Note that we have to restrict to a Zariski open $\mathcal{U} \subset \mathcal{U}'$ since theorem 15 only works for a generic geometric square for which the corresponding \mathbb{P}^1 's do not intersect. Also, \mathbb{H} is a member of the family by example 25. \square

Example 27. Consider the type-A cubic algebra from example 22 for the parameters $(0 : 1 : 1)$. In this case the matrix describing ϕ_w is the identity matrix, hence the two \mathbb{P}^1 's coincide and theorem 15 does not apply.

4 Further remarks

Based on the result for \mathbb{P}^2 from [18] Orlov conjectured informally that every noncommutative deformation can be embedded in some commutative deformation, i.e. for every smooth projective variety X there exists a smooth projective variety Y and a fully faithful functor $\mathbf{D}^b(X) \hookrightarrow \mathbf{D}^b(Y)$ such that for every noncommutative deformation of X there is a commutative deformation of Y such that there is again a fully faithful functor between the bounded derived categories.

The result in this paper adds some further evidence to this, by proving the result for $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $Y = \text{Hilb}^2 \mathbb{P}^1 \times \mathbb{P}^1$. The general construction from [18] seems to prove this conjecture in case $\mathbf{D}^b(X)$ has a full and strong exceptional collection: noncommutative deformations of X correspond to changing the relations in the quiver, and these changes are reflected by changing the vector bundles in the iterated projective bundle construction.

However, it would be interesting to know whether one can always choose for Y a natural moduli space associated to X , as is the case for \mathbb{P}^2 and $\mathbb{P}^1 \times \mathbb{P}^1$ where one can take the Hilbert scheme of two points. To investigate this in a more general setting we formulate an infinitesimal version of this conjecture in terms of limited functoriality for Hochschild cohomology, and explain how results on Poisson structures on surfaces give some substance to this conjecture in special cases.

The infinitesimal deformation theory of abelian categories is governed by their Hochschild cohomology [13], and one has the Hochschild–Kostant–Rosenberg de-

composition for Hochschild cohomology of smooth varieties. In particular there is the decomposition

$$\mathrm{HH}^2(X) = \mathrm{H}^0(X, \bigwedge^2 \mathcal{T}_X) \oplus \mathrm{H}^1(X, \mathcal{T}_X) \oplus \mathrm{H}^2(X, \mathcal{O}_X) \quad (79)$$

where the first term can be understood as the noncommutative deformations, the second as the commutative (or geometric) deformations and the third one corresponding to gerby deformations [21, 4].

The natural categorical framework for Hochschild cohomology is that of dg categories. It is easily checked that Hochschild cohomology is not functorial for arbitrary functors: it only satisfies a limited functoriality. Indeed, in the case of a dg functor inducing a fully faithful embedding on the level of derived categories there is an induced morphism on the Hochschild cohomologies [10], which in the case of Fourier–Mukai transforms is treated in [12].

Combining limited functoriality with the Hochschild–Kostant–Rosenberg decomposition one could formulate an infinitesimal version of Orlov’s conjecture as follows.

Question 28. Let X be a smooth projective variety. Does there exist a smooth projective variety Y and a fully faithful embedding $\mathbf{D}^b(X) \rightarrow \mathbf{D}^b(Y)$, such that the induced morphism on Hochschild cohomologies induces a surjective morphism

$$\mathrm{H}^1(Y, \mathcal{T}_Y) \twoheadrightarrow \mathrm{H}^0(X, \bigwedge^2 \mathcal{T}_X). \quad (80)$$

Sadly, we do not even know the answer for the embeddings obtained for $X = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ and $Y = \mathrm{Hilb}^2 X$.

Some positive evidence comes from a result by Hitchin who shows in [8] the existence of the split exact sequence

$$0 \rightarrow \mathrm{H}^1(S, \mathcal{T}_S) \rightarrow \mathrm{H}^1(\mathrm{Hilb}^n S, \mathcal{T}_{\mathrm{Hilb}^n S}) \rightarrow \mathrm{H}^0(S, \omega_S^\vee) \rightarrow 0 \quad (81)$$

where S is a smooth projective surface over the complex numbers.

Again one does not know that the morphism on the right is related to (80), but it does show that a possible approach might be to choose for Y a smooth projective variety representing a moduli problem associated to X . The choice of the Hilbert scheme of n points seems to be a natural choice in the case of a surface, but for higher-dimensional varieties the Hilbert scheme fails to be smooth in general.

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